PAIRS OF COMMUTING ISOMETRIES - I

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ABSTRACT. We present an explicit version of Berger, Coburn and Lebow's classification result for pure pairs of commuting isometries in the sense of an explicit recipe for constructing pairs of commuting isometric multipliers with precise coefficients. We describe a complete set of (joint) unitary invariants and compare the Berger, Coburn and Lebow's representations with other natural analytic representations of pure pairs of commuting isometries. Finally, we study the defect operators of pairs of commuting isometries.

1. INTRODUCTION

A very general and fundamental problem in the theory of bounded linear operators on Hilbert spaces is to find classifications and representations of commuting families of isometries.

In the case of single isometries this question has a complete and explicit answer: If V is an isometry on a Hilbert space $\mathcal{H}$, then there exist a Hilbert space $\mathcal{H}_u$ and a unitary operator U on $\mathcal{H}_u$ such that V on $\mathcal{H}$ and $\begin{bmatrix} S \otimes I_{\mathcal{W}} & 0 \\ 0 & U \end{bmatrix}$ on $(l^2(\mathbb{Z}_+) \otimes \mathcal{W}) \oplus \mathcal{H}_u$ are unitarily equivalent, where $\mathcal{W} = \ker V^*$ is the wandering subspace for V and S is the forward shift operator on $l^2(\mathbb{Z}_+)$ [H]. This fundamental result is due to J. von Neumann [VN] and H. Wold [W] (see Theorem 2.1 for more details).

The case of pairs (and n-tuples) of commuting isometries is more subtle, and is directly related to the commutant lifting theorem [FF] (in terms of an explicit, and then unique solution), invariant subspace problem [HH] and representations of contractions on Hilbert spaces in function Hilbert spaces [NF]. For instance:

(a) Let $\mathcal{S}$ be a closed joint (M_{z_1}, M_{z_2})-invariant subspace of the Hardy space $H^2(\mathbb{D}^2)$. Then $(M_{z_1}|_{\mathcal{S}}, M_{z_2}|_{\mathcal{S}})$ on $\mathcal{S}$ is a pure (see Section 3) pair of commuting isometries. Classification of such pairs of isometries is largely unknown (see Rudin [R]).

(b) Let T be a contraction on a Hilbert space $\mathcal{H}$. Then there exists a pair of commuting isometries (V_1, V_2) on a Hilbert space $\mathcal{K}$ such that T and $P_{\ker V_2^*} V_1|_{\ker V_2^*}$ are unitarily equivalent (see Bercovici, Douglas and Foias [BDF]).

(c) The celebrated Ando dilation theorem (see Ando [A]) states that a commuting pair of contractions dilates to a commuting pair of isometries. Again, the structure of Ando's pairs of commuting isometries is largely unknown.

The main purpose of this paper is to explore and relate various natural representations of a large class of pairs of commuting isometries on Hilbert spaces. The geometry of Hilbert spaces, the classical Wold-von Neumann decomposition for isometries, the analytic structure of the commutator of the unilateral shift, and the Berger, Coburn and Lebow [BCL] representations of pure pairs of commuting isometries are the main guiding principles for our study. The Berger, Coburn and Lebow theorem states that: Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$, and let $V = V_1 V_2$ be a shift (or, a pure isometry - see Section 2). Then there exist a Hilbert space $\mathcal{W}$, an orthogonal projection P and a unitary operator U on $\mathcal{W}$ such that

$$\Phi_1(z) = U^*(P + zP^\perp) \quad \text{and} \quad \Phi_2(z) = (P^\perp + zP)U \qquad (z \in \mathbb{D}),$$

are commuting isometric multipliers in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$, and (V_1, V_2, V) on $\mathcal{H}$ and $(M_{\Phi_1}, M_{\Phi_2}, M_z)$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent (see Bercovici, Douglas and Foias [BDF] for an elegant proof).

Here and further on, given a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$. We also set

$$P_{\mathcal{S}}^\perp = I_{\mathcal{H}} - P_{\mathcal{S}}.$$

In this paper we give a new and more concrete treatment, in the sense of explicit representations and analytic descriptions, to the structure of pure pairs of commuting isometries. More specifically, we provide an explicit recipe for constructing the isometric multipliers $(\Phi_1(z), \Phi_2(z))$, and the operators U and P involved in the coefficients of Φ_1 and Φ_2 (see Theorems 3.2 and 3.3). Then we compare the Berger, Coburn and Lebow representations with other possible analytic representations of pairs of commuting isometries.

In Section 6, which is independent of the remaining part of the paper, we analyze defect operators for (not necessarily pure) pairs of commuting isometries. We provide a list of characterizations of pairs of commuting

isometries with positive defect operators (see Theorem 6.2). Our results hold in a more general setting with somewhat simpler proofs (see Theorem 6.5 for instance) than the one considered by He, Qin and Yang [HQY]. Moreover, we prove that for a large class of pure pairs of commuting isometries the defect operator is negative if and only if the defect operator is the zero operator.

The paper is organized as follows. In Section 2 we review the classical Wold-von Neumann theorem for isometries and then prove a representation theorem for commutators of shifts. In Section 3 we discuss some basic relationships between wandering subspaces for commuting isometries, followed by a new and explicit proof of the Berger, Coburn and Lebow characterizations of pure pairs of commuting isometries. Section 4 is devoted to a short discussion about joint unitary invariants of pure pairs of commuting isometries. Section 5 ties together the explicit Berger, Coburn and Lebow representation and other possible analytic representations of a pair of commuting isometries. Then, in Section 6, we present a general theory for pairs of commuting isometries and analyze the defect operators. Concluding remarks, future directions and a close connection of our consideration with the Sz.-Nagy and Foias characteristic functions for contractions are discussed in Section 7.

2. Wold-von Neumann decomposition and commutators

We begin this section by briefly recalling the construction of the classical Wold-von Neumann decomposition of isometric operators on Hilbert spaces. Here our presentation is more algebraic and geared towards the main theme of the paper. First, recall that an isometry V on a Hilbert space $\mathcal{H}$ is said to be *pure*, or a *shift*, if it has no unitary direct summand, or equivalently, if $\lim_{m \to \infty} V^{*m} = 0$ in the strong operator topology (see Halmos [H]).

Let V be an isometry on a Hilbert space $\mathcal{H}$, and let $\mathcal{W}(V)$ be the *wandering subspace* [H] for V, that is,

$$\mathcal{W}(V) = \mathcal{H} \ominus V\mathcal{H}.$$

The classical Wold-von Neumann decomposition is as follows:

Theorem 2.1. *(Wold-von Neumann decomposition) Let V be an isometry on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ decomposes as a direct sum of V-reducing subspaces $\mathcal{H}_s(V) = \bigoplus_{m=0}^{\infty} V^m \mathcal{W}(V)$ and $\mathcal{H}_u(V) = \mathcal{H} \ominus \mathcal{H}_s(V)$ and*

$$V = \begin{bmatrix} V_s & 0 \\ 0 & V_u \end{bmatrix} \in \mathcal{B}(\mathcal{H}_s(V) \oplus \mathcal{H}_u(V)),$$

where $V_s = V|_{\mathcal{H}_s(V)}$ is a shift operator and $V_u = V|_{\mathcal{H}_u(V)}$ is a unitary operator.

We will refer to this decomposition as the *Wold-von Neumann orthogonal decomposition* of V.

Recall that the *Hardy space* $H^2(\mathbb{D})$ is the Hilbert space of all analytic functions on the unit disc $\mathbb{D}$ with square summable Taylor coefficients (cf. [H], [RR]). The Hardy space is also a reproducing kernel Hilbert space corresponding to the Szegö kernel

$$\mathbb{S}(z, w) = (1 - z\bar{w})^{-1} \qquad (z, w \in \mathbb{D}).$$

For any Hilbert space $\mathcal{E}$, the $\mathcal{E}$-valued Hardy space with reproducing kernel

$$\mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E}),\ (z, w) \mapsto \mathbb{S}(z, w)I_{\mathcal{E}},$$

can canonically be identified with the tensor product Hilbert space $H^2(\mathbb{D}) \otimes \mathcal{E}$. To simplify the notation, we often identify $H^2(\mathbb{D}) \otimes \mathcal{E}$ with the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$. The space of $\mathcal{B}(\mathcal{E})$-valued bounded holomorphic functions on $\mathbb{D}$ will be denoted by $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$.
Let $M^{\mathcal{E}}_z$ denote the multiplication operator by the coordinate function z on $H^2_{\mathcal{E}}(\mathbb{D})$, that is

$$(M^{\mathcal{E}}_z f)(w) = wf(w) \qquad (f \in H^2_{\mathcal{E}}(\mathbb{D}), w \in \mathbb{D}).$$

Then $M^{\mathcal{E}}_z$ is a shift operator and

$$\mathcal{W}(M^{\mathcal{E}}_z) = \mathcal{E}.$$

To simplify the notation we often omit the superscript and denote $M^{\mathcal{E}}_z$ by M_z, if $\mathcal{E}$ is clear from the context.

We now proceed to give an analytic description of the Wold-von Neumann construction.

Let V be an isometry on $\mathcal{H}$, and let $\mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V)$ be the Wold-von Neumann orthogonal decomposition of V. Define

$$\Pi_V : \mathcal{H}_s(V) \oplus \mathcal{H}_u(V) \to H^2_{\mathcal{W}(V)}(\mathbb{D}) \oplus \mathcal{H}_u(V)$$

by

$$\Pi_V(V^m \eta \oplus f) = z^m \eta \oplus f \qquad (m \geq 0, \eta \in \mathcal{W}(V), f \in \mathcal{H}_u(V)).$$

Then Π_V is a unitary and

$$\Pi_V \begin{bmatrix} V_s & 0 \\ 0 & V_u \end{bmatrix} = \begin{bmatrix} M^{\mathcal{W}(V)}_z & 0 \\ 0 & V_u \end{bmatrix} \Pi_V.$$

In particular, if V is a shift, then $\mathcal{H}_u(V) = \{0\}$ and hence

$$\Pi_V V = M^{\mathcal{W}(V)}_z \Pi_V.$$

Therefore, an isometry V on $\mathcal{H}$ is a shift operator if and only if V is unitarily equivalent to $M_z^{\mathcal{E}}$ on $H_{\mathcal{E}}^2(\mathbb{D})$, where $\dim \mathcal{E} = \dim \mathcal{W}(V)$.

In the sequel we denote by $(\Pi_V, M_z^{\mathcal{W}(V)})$, or simply by (Π_V, M_z), the Wold-von Neumann decomposition of the pure isometry V in the above sense.

Let $\mathcal{E}$ be a Hilbert space, and let C be a bounded linear operator on $H_{\mathcal{E}}^2(\mathbb{D})$. Then $C \in \{M_z\}'$, that is, $CM_z = M_zC$, if and only if (cf. [NF])

$$C = M_\Theta$$

for some $\Theta \in H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D})$ and $(M_\Theta f)(w) = \Theta(w)f(w)$ for all $f \in H_{\mathcal{E}}^2(\mathbb{D})$ and $w \in \mathbb{D}$.

Now let V be a pure isometry, and let $C \in \{V\}'$. Let (Π_V, M_z) be the Wold-von Neumann decomposition of V, and let $\mathcal{W} = \mathcal{W}(V)$. Since $\Pi_V C \Pi_V^*$ on $H_{\mathcal{W}}^2(\mathbb{D})$ is the representation of C on $\mathcal{H}$ and $(\Pi_V C \Pi_V^*)M_z = M_z(\Pi_V C \Pi_V^*)$, it follows that

$$\Pi_V C \Pi_V^* = M_\Theta,$$

for some $\Theta \in H_{\mathcal{B}(\mathcal{W})}^\infty(\mathbb{D})$. The main result of this section is the following explicit representation of Θ.

Theorem 2.2. *Let V be a pure isometry on $\mathcal{H}$, and let C be a bounded operator on $\mathcal{H}$. Let (Π_V, M_z) be the Wold-von Neumann decomposition of V. Set $\mathcal{W} = \mathcal{W}(V)$, $M = \Pi_V C \Pi_V^*$ and let*

$$\Theta(w) = P_{\mathcal{W}}(I_{\mathcal{H}} - wV^*)^{-1} C \mid_{\mathcal{W}} \qquad (w \in \mathbb{D}).$$

Then

$$CV = VC,$$

if and only if $\Theta \in H_{\mathcal{B}(\mathcal{W})}^\infty(\mathbb{D})$ and

$$M = M_\Theta.$$

Proof. Let $h \in \mathcal{H}$. One can express h as $h = \sum_{m=0}^{\infty} V^m \eta_m$, for some $\eta_m \in \mathcal{W}$, $m \geq 0$ (as $\mathcal{H} = \overset{\infty}{\underset{m=0}{\oplus}} V^m \mathcal{W}$). Applying $P_{\mathcal{W}} V^{*l}$ to both sides and using the fact that $\mathcal{W} = \mathcal{W}(V) = \ker V^*$, we obtain $\eta_l = P_{\mathcal{W}} V^{*l} h$ for all $l \geq 0$. This implies, for any $h \in \mathcal{H}$,

$$(2.1) \qquad h = \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} h.$$

Now let $CV = VC$. Then there exists a bounded analytic function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ such that $\Pi_V C \Pi_V^* = M_\Theta$. For each $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$ we have

$$
\begin{aligned}
\Theta(w)\eta &= (M_\Theta \eta)(w) \\
&= (\Pi_V C \Pi_V^* \eta)(w) \\
&= (\Pi_V C \eta)(w),
\end{aligned}
$$

as $\Pi_V^* \eta = \eta$. Since in view of (2.1)

$$
C\eta = \sum_{m=0}^\infty V^m P_{\mathcal{W}} V^{*m} C\eta,
$$

it follows that

$$
\begin{aligned}
\Theta(w)\eta &= (\Pi_V (\sum_{m=0}^\infty V^m P_{\mathcal{W}} V^{*m} C\eta))(w) \\
&= (\sum_{m=0}^\infty M_z^m (P_{\mathcal{W}} V^{*m} C\eta))(w) \\
&= \sum_{m=0}^\infty w^m (P_{\mathcal{W}} V^{*m} C\eta) \\
&= P_{\mathcal{W}} (I_{\mathcal{H}} - wV^*)^{-1} C\eta.
\end{aligned}
$$

Therefore

$$
\Theta(w) = P_{\mathcal{W}} (I_{\mathcal{H}} - wV^*)^{-1} C|_{\mathcal{W}} \qquad (w \in \mathbb{D}),
$$

as required. Finally, since the sufficient part is trivial, the proof is complete. ∎

Note that in the above proof we have used the standard projection formula (see, for example, Rosenblum and Rovnyak [RR]) $I_{\mathcal{H}} = \text{SOT} - \sum_{m=0}^\infty V^m P_{\mathcal{W}} V^{*m}$. It may also be observed that $\|wV^*\| = |w|\|V\| < 1$ for all $w \in \mathbb{D}$, and so it follows that the function Θ defined in Theorem 2.2 is a $\mathcal{B}(\mathcal{W})$-valued holomorphic function in the unit disc $\mathbb{D}$. However, what is not guaranteed in general here is that the function Θ is in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. The above theorem says that this is so if and only if $CV = VC$.

3. Berger, Coburn and Lebow representations

This section is devoted to a detailed study of Berger, Coburn and Lebow's representation of pure pairs of commuting isometries. Our approach is different and yields sharper results, along with new proofs, in terms of explicit coefficients of the one variable polynomials associated with the class of pure pairs of commuting isometries. Before dealing more specifically with pure

pairs of commuting isometries we begin with some general observations about pairs of commuting isometries.

Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. In the sequel, we will adopt the following notations:

$$V = V_1 V_2,$$

$$\mathcal{W} = \mathcal{W}(V) = \mathcal{W}(V_1 V_2) = \mathcal{H} \ominus V_1 V_2 \mathcal{H},$$

and

$$\mathcal{W}_j = \mathcal{W}(V_j) = \mathcal{H} \ominus V_j \mathcal{H} \qquad (j = 1, 2).$$

A pair of commuting isometries (V_1, V_2) on $\mathcal{H}$ is said to be *pure* if V is a pure isometry.

The following useful lemma on wandering subspaces for commuting isometries is simple.

Lemma 3.1. *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then*

$$\mathcal{W} = \mathcal{W}_1 \oplus V_1 \mathcal{W}_2 = V_2 \mathcal{W}_1 \oplus \mathcal{W}_2,$$

and the operator U on $\mathcal{W}$ defined by

$$U(\eta_1 \oplus V_1 \eta_2) = V_2 \eta_1 \oplus \eta_2,$$

for $\eta_1 \in \mathcal{W}_1$ and $\eta_2 \in \mathcal{W}_2$, is a unitary operator. Moreover,

$$P_{\mathcal{W}} V_i = V_i P_{\mathcal{W}_j} \qquad (i \neq j).$$

Proof. The first equality follows from

$$I - VV^* = (I - V_1 V_1^*) \oplus V_1 (I - V_2 V_2^*) V_1^* = V_2 (I - V_1 V_1^*) V_2^* \oplus (I - V_2 V_2^*).$$

The second part directly follows from the first part, and the last claim follows from $(I - VV^*)V_i = V_i(I - V_j V_j^*)$ for all $i \neq j$. This concludes the proof of the lemma. ∎

Let (V_1, V_2) be a pure pair of commuting isometries on a Hilbert space $\mathcal{H}$, and let (Π_V, M_z) be the Wold-von Neumann decomposition of V. Since

$$VV_i = V_i V \qquad (i = 1, 2),$$

there exist isometric multipliers (that is, inner functions [NF]) Φ_1 and Φ_2 in $H_{\mathcal{B}(\mathcal{W})}^\infty(\mathbb{D})$ such that

$$\Pi_V V_i = M_{\Phi_i} \Pi_V \qquad (i = 1, 2).$$

In other words, (M_{Φ_1}, M_{Φ_2}) on $H_{\mathcal{W}}^2(\mathbb{D})$ is the representation of (V_1, V_2) on $\mathcal{H}$. Following Berger, Coburn and Lebow [BCL], we say that (M_{Φ_1}, M_{Φ_2}) is the *BCL representation* of (V_1, V_2), or simply the *BCL pair* corresponding to (V_1, V_2).

We now present an explicit description of the BCL pair (M_{Φ_1}, M_{Φ_2}).

Theorem 3.2. *Let (V_1, V_2) be a pure pair of commuting isometries on a Hilbert space $\mathcal{H}$, and let (M_{Φ_1}, M_{Φ_2}) be the BCL representation of (V_1, V_2). Then*

$$\Phi_1(z) = V_1|_{\mathcal{W}_2} \oplus V_2^*|_{V_2 \mathcal{W}_1} z, \;\; \Phi_2(z) = V_2|_{\mathcal{W}_1} \oplus V_1^*|_{V_1 \mathcal{W}_2} z,$$

for all $z \in \mathbb{D}$.

Proof. Let η in $\mathcal{W} = V_2 \mathcal{W}_1 \oplus \mathcal{W}_2$, and let $w \in \mathbb{D}$. Then there exist $\eta_1 \in \mathcal{W}_1$ and $\eta_2 \in \mathcal{W}_2$ such that $\eta = V_2 \eta_1 \oplus \eta_2$. Then $V_1 \eta = V \eta_1 + V_1 \eta_2$, and hence

$$\Phi_1(w)\eta = (M_{\Phi_1}\eta)(w) = (\Pi_V V_1 \Pi_V^* \eta)(w) = (\Pi_V V_1 \eta)(w) = (\Pi_V V \eta_1 + \Pi_V V_1 \eta_2)(w).$$

This along with the fact that $V_1 \eta_2 \in \mathcal{W}$ (see Lemma 3.1) gives

$$\begin{aligned}
\Phi_1(w)\eta &= (M_z \Pi_V \eta_1 + V_1 \eta_2)(w) \\
&= (M_z \eta_1 + V_1 \eta_2)(w) \\
&= w\eta_1 + V_1 \eta_2 \\
&= wV_2^* \eta + V_1 \eta_2,
\end{aligned}$$

for all $w \in \mathbb{D}$. Therefore

$$\Phi_1(z) = V_1|_{\mathcal{W}_2} \oplus V_2^*|_{V_2 \mathcal{W}_1} z,$$

for all $z \in \mathbb{D}$, as $\mathcal{W}_2 = Ker(V_2^*)$. The representation of Φ_2 follows similarly.
∎

In the following, we present Berger, Coburn and Lebow's version of representations of pure pairs of commuting isometries. This yields an explicit representations of the auxiliary operators U and P (see Section 1). The proof readily follows from Lemma 3.1 and Theorem 3.2.

Theorem 3.3. *Let (V_1, V_2) be a pure pair of commuting isometries on $\mathcal{H}$. Then the BCL pair (M_{Φ_1}, M_{Φ_2}) corresponding to (V_1, V_2) is given by*

$$\Phi_1(z) = U^*(P_{\mathcal{W}_2} + zP_{\mathcal{W}_2}^\perp),$$

and

$$\Phi_2(z) = (P_{\mathcal{W}_2}^\perp + zP_{\mathcal{W}_2})U,$$

where

$$U = \begin{bmatrix} V_2|_{\mathcal{W}_1} & 0 \\ 0 & V_1^*|_{V_1 \mathcal{W}_2} \end{bmatrix} : \begin{array}{c} \mathcal{W}_1 \\ \oplus \\ V_1 \mathcal{W}_2 \end{array} \rightarrow \begin{array}{c} V_2 \mathcal{W}_1 \\ \oplus \\ \mathcal{W}_2 \end{array},$$

is a unitary operator on $\mathcal{W}$.

Therefore, $(V_1, V_2, V_1 V_2)$ on $\mathcal{H}$ and $(M_{\Phi_1}, M_{\Phi_2}, M_z^{\mathcal{W}})$ on $H_{\mathcal{W}}^2(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W}$ is the wandering subspace for $V = V_1 V_2$.

4. Unitary invariants

In this short section we present a complete set of joint unitary invariants for pure pairs of commuting isometries. Recall that two commuting pairs (T_1, T_2) and $(\tilde{T}_1, \tilde{T}_2)$ on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, are said to be (jointly) unitarily equivalent if there exists a unitary operator $U : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $UT_j = \tilde{T}_j U$ for all $j = 1, 2$.

First we note that, by virtue of Theorem 2.9 of [BDF], the orthogonal projection $P_{\mathcal{W}_2}$ and the unitary operator U on $\mathcal{W}$, as in Theorem 3.3, form a complete set of (joint) unitary invariants of pure pairs of commuting isometries. More specifically: Let (V_1, V_2) and $(\tilde{V}_1, \tilde{V}_2)$ be two pure pairs of commuting isometries on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Let $\tilde{\mathcal{W}}_j$ be the wandering subspace for $\tilde{V}_j$, $j = 1, 2$. Then (V_1, V_2) and $(\tilde{V}_1, \tilde{V}_2)$ are unitarily equivalent if and only if

$$
\left(\begin{bmatrix} V_2|_{\mathcal{W}_1} & 0 \\ 0 & V_1^*|_{V_1 \mathcal{W}_2} \end{bmatrix}, P_{\mathcal{W}_2} \right) \quad \text{and} \quad \left(\begin{bmatrix} \tilde{V}_2|_{\tilde{\mathcal{W}}_1} & 0 \\ 0 & \tilde{V}_1^*|_{\tilde{V}_1 \tilde{\mathcal{W}}_2} \end{bmatrix}, P_{\tilde{\mathcal{W}}_2} \right)
$$

are unitarily equivalent.

In addition to the above, the following unitary invariants are also explicit. The proof is an easy consequence of Theorem 3.2. Here we will make use of the identifications of A on $H_{\mathcal{W}}^2(\mathbb{D})$ and AM_z on $H_{\mathcal{W}}^2(\mathbb{D})$ with $I_{H^2(\mathbb{D})} \otimes A$ on $H^2(\mathbb{D}) \otimes \mathcal{W}$ and $M_z \otimes A$ on $H^2(\mathbb{D}) \otimes \mathcal{W}$, respectively, where $A \in \mathcal{B}(\mathcal{W})$ (see Section 2).

Theorem 4.1. *Let (V_1, V_2) and $(\tilde{V}_1, \tilde{V}_2)$ be two pure pairs of commuting isometries on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Then (V_1, V_2) and $(\tilde{V}_1, \tilde{V}_2)$ are unitarily equivalent if and only if $(V_1|_{\mathcal{W}_2}, V_2^*|_{V_2 \mathcal{W}_1})$ and $(\tilde{V}_1|_{\tilde{\mathcal{W}}_2}, \tilde{V}_2^*|_{\tilde{V}_2 \tilde{\mathcal{W}}_1})$ are unitarily equivalent.*

Proof. Let (M_{Φ_1}, M_{Φ_2}) and $(M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})$ be the BCL pairs corresponding to (V_1, V_2) and $(\tilde{V}_1, \tilde{V}_2)$, respectively, as in Theorem 3.2. Let $C_1 = V_1|_{\mathcal{W}_2}$ and $C_2 = V_2^*|_{V_2 \mathcal{W}_1}$ be the coefficients of Φ_1. Similarly, let $\tilde{C}_1$ and $\tilde{C}_2$ be the coefficients of $\tilde{\Phi}_1$.

Now let $Z : \mathcal{W} \to \tilde{\mathcal{W}}$ be a unitary such that $ZC_j = \tilde{C}_j Z$, $j = 1, 2$. Then

$$
\begin{aligned}
M_{\Phi_1} &= I_{H^2(\mathbb{D})} \otimes C_1 + M_z \otimes C_2 \\
&= I_{H^2(\mathbb{D})} \otimes Z^* \tilde{C}_1 Z + M_z \otimes Z^* \tilde{C}_2 Z \\
&= (I_{H^2(\mathbb{D})} \otimes Z^*)(I_{H^2(\mathbb{D})} \otimes \tilde{C}_1 + M_z \otimes \tilde{C}_2)(I_{H^2(\mathbb{D})} \otimes Z) \\
&= (I_{H^2(\mathbb{D})} \otimes Z^*) M_{\tilde{\Phi}_1} (I_{H^2(\mathbb{D})} \otimes Z).
\end{aligned}
$$

Because $M_{\Phi_2} = M_z M_{\Phi_1}^*$ and $M_{\tilde{\Phi}_2} = M_z M_{\tilde{\Phi}_1}^*$, it follows that (M_{Φ_1}, M_{Φ_2}) and $(M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})$ are unitarily equivalent, that is, (V_1, V_2) and $(\tilde{V}_1, \tilde{V}_2)$ are unitarily equivalent.

To prove the necessary part, let (M_{Φ_1}, M_{Φ_2}) and $(M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})$ are unitarily equivalent. Then there exists a unitary operator $X : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ [RR] such that

$$XM_{\Phi_j} = M_{\tilde{\Phi}_j} X \qquad (j = 1, 2).$$

Since

$$XM_z^{\mathcal{W}} = XM_{\Phi_1}M_{\Phi_2} = M_{\tilde{\Phi}_1}XX^*M_{\tilde{\Phi}_2}X = M_{\tilde{\Phi}_1}M_{\tilde{\Phi}_2}X = M_z^{\tilde{\mathcal{W}}}X,$$

there exists a unitary operator $Z : \mathcal{W} \to \tilde{\mathcal{W}}$ such that

$$X = I_{H^2(\mathbb{D})} \otimes Z.$$

This and $XM_{\Phi_1} = M_{\tilde{\Phi}_1}X$ implies that

$$(I_{H^2(\mathbb{D})} \otimes Z)(I_{H^2(\mathbb{D})} \otimes C_1 + M_z \otimes C_2) = (I_{H^2(\mathbb{D})} \otimes \tilde{C}_1 + M_z \otimes \tilde{C}_2)(I_{H^2(\mathbb{D})} \otimes Z).$$

Hence (C_1, C_2) and $(\tilde{C}_1, \tilde{C}_2)$ are unitarily equivalent. This completes the proof of the theorem. ∎

Observe that the set of joint unitary invariants $\{V_1|_{\mathcal{W}_2}, V_2^*|_{V_2\mathcal{W}_1}\}$, as above, is associated with the coefficients of Φ_1 of the BCL pair (M_{Φ_1}, M_{Φ_2}) corresponding to (V_1, V_2). Clearly, by duality, a similar statement holds for the coefficients of Φ_2 as well: $\{V_2|_{\mathcal{W}_1}, V_1^*|_{V_1\mathcal{W}_2}\}$ is a complete set of joint unitary invariants for pure pairs of commuting isometries.

5. Pure isometries

In this section we will analyze pairs of commuting isometries (V_1, V_2) such that either V_1 or V_2 is a pure isometry, or both V_1 and V_2 are pure isometries. We begin with a concrete example which illustrates this particular class and also exhibits its complex structure.

Recall that the Hardy space $H^2(\mathbb{D}^2)$ over the bidisc $\mathbb{D}^2$ is the Hilbert space of all analytic functions on the bidisc $\mathbb{D}^2$ with square summable Taylor coefficients (see Rudin [R]). Let M_{z_j} on $H^2(\mathbb{D}^2)$ be the multiplication operator by the coordinate function z_j, $j = 1, 2$. Note that (M_{z_1}, M_{z_2}) on $H^2(\mathbb{D}^2)$ can be identified with $(M_z \otimes I_{H^2(\mathbb{D})}, I_{H^2(\mathbb{D})} \otimes M_z)$ on $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$, and consequently, (M_{z_1}, M_{z_2}) on $H^2(\mathbb{D}^2)$ is a pair of *doubly commuting* (that is, $M_{z_1}^* M_{z_2} = M_{z_2} M_{z_1}^*$) pure isometries.

Now let $\mathcal{S}$ be a joint (M_{z_1}, M_{z_2})-invariant closed subspace of $H^2(\mathbb{D}^2)$, that is, $M_{z_j}\mathcal{S} \subseteq \mathcal{S}$. Set

$$V_j = M_{z_j}|_{\mathcal{S}} \qquad (j = 1, 2).$$

It follows immediately that V_j is a pure isometry and $V_1 V_2 = V_2 V_1$, and hence (V_1, V_2) is a pair of commuting pure isometries on $\mathcal{S}$.

If we assume, in addition, that (V_1, V_2) is doubly commuting (that is, $V_1^* V_2 = V_2 V_1^*$), then it follows that (V_1, V_2) on $\mathcal{S}$ and (M_{z_1}, M_{z_2}) on $H^2(\mathbb{D}^2)$ are unitarily equivalent. See Slocinski [S] for more details. In general, however, the classification of pairs of commuting isometries, up to unitary equivalence, is complicated and very little seems to be known. For instance, see Rudin [R] for a list of pathological examples (also see Qin and Yang [QY]).

We now turn our attention to the general problem. Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$, and let V_1 be a pure isometry. Then, in particular, $V = V_1 V_2$ is a pure isometry, and hence (V_1, V_2) is a pure pair of commuting isometries. Since $V_1 V_2 = V_2 V_1$, by Theorem 2.2, it follows that

$$(5.1) \qquad \Pi_{V_1} V_2 = M_{\Theta_{V_2}} \Pi_{V_1},$$

where $\Theta_{V_2} \in H^\infty_{\mathcal{B}(\mathcal{W}_1)}(\mathbb{D})$ is an inner multiplier and

$$(5.2) \qquad \Theta_{V_2}(z) = P_{\mathcal{W}_1}(I_{\mathcal{H}} - z V_1^*)^{-1} V_2|_{\mathcal{W}_1} \qquad (z \in \mathbb{D}).$$

Let (M_{Φ_1}, M_{Φ_2}) be the BCL pair (see Theorem 3.3) corresponding to (V_1, V_2), that is, $\Pi_V V_i = M_{\Phi_i} \Pi_V$ for all $i = 1, 2$. Set

$$\tilde{\Pi}_1 = \Pi_{V_1} \Pi_V^*.$$

Then $\tilde{\Pi}_1 : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{W}_1}(\mathbb{D})$ is a unitary operator such that $\tilde{\Pi}_1 M_{\Phi_1} = M_z^{\mathcal{W}_1} \tilde{\Pi}_1$ and $\tilde{\Pi}_1 M_{\Phi_2} = M_{\Theta_{V_2}} \tilde{\Pi}_1$. Therefore, we have the following commutative diagram:

$$\mathcal{H} \xrightarrow{\Pi_V} H^2_{\mathcal{W}}(\mathbb{D})$$
$$\Pi_{V_1} \searrow \qquad \downarrow \tilde{\Pi}_1$$
$$H^2_{\mathcal{W}_1}(\mathbb{D})$$

where (M_{Φ_1}, M_{Φ_2}) on $H^2_{\mathcal{W}}(\mathbb{D})$ and $(M_z^{\mathcal{W}_1}, M_{\Theta_{V_2}})$ on $H^2_{\mathcal{W}_1}(\mathbb{D})$ are the representations of (V_1, V_2) on $\mathcal{H}$.

We now proceed to settle the non-trivial part of this consideration: An analytic description of the unitary map $\tilde{\Pi}_1$. To this end, observe first that since $\Pi_{V_1} V_1 = M_z^{\mathcal{W}_1} \Pi_{V_1}$, (5.1) gives

$$\Pi_{V_1} V = M_z^{\mathcal{W}_1} M_{\Theta_{V_2}} \Pi_{V_1}.$$

Then

$$\tilde{\Pi}_1 M_z^{\mathcal{W}} = \Pi_{V_1} V \Pi_V^* = M_z^{\mathcal{W}_1} M_{\Theta_{V_2}} \Pi_{V_1} \Pi_V^*,$$

that is,

$$(5.3) \qquad \tilde{\Pi}_1 M_z^{\mathcal{W}} = (M_z^{\mathcal{W}_1} M_{\Theta_{V_2}}) \tilde{\Pi}_1.$$

Let $\eta \in \mathcal{W}$. By Equation (2.1) we can write $\eta = \sum_{m=0}^{\infty} V_1^m P_{\mathcal{W}_1} V_1^{*m} \eta$. Therefore

$$(\Pi_{V_1} \eta)(w) = (\sum_{m=0}^{\infty} \Pi_{V_1} V_1^m P_{\mathcal{W}_1} V_1^{*m} \eta)(w)$$

$$= (\sum_{m=0}^{\infty} M_z^m P_{\mathcal{W}_1} V_1^{*m} \eta)(w)$$

$$= \sum_{m=0}^{\infty} w^m (P_{\mathcal{W}_1} V_1^{*m} \eta),$$

which yields

$$\tilde{\Pi}_1 \eta = \Pi_{V_1} \Pi_V^* \eta = \Pi_{V_1} \eta = \sum_{m=0}^{\infty} z^m (P_{\mathcal{W}_1} V_1^{*m} \eta),$$

that is

$$\tilde{\Pi}_1 \eta = P_{\mathcal{W}_1} [I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1} V_1^*] \eta,$$

for all $\eta \in \mathcal{W}$. It now follows from (5.3) that

$$\tilde{\Pi}_1(z^m \eta) = (z\Theta_{V_2}(z))^m P_{\mathcal{W}_1} [I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1} V_1^*] \eta,$$

for all $m \geq 0$, and so, by $\mathbb{S}(\cdot, w)\eta = \sum_{m=0}^{\infty} z^m \bar{w}^m \eta$, it follows that

$$\tilde{\Pi}_1(\mathbb{S}(\cdot, w)\eta) = \tilde{\Pi}_1(\sum_{m=0}^{\infty} z^m \bar{w}^m \eta)$$

$$= (I_{\mathcal{W}_1} - \bar{w}z\Theta_{V_2}(z))^{-1} P_{\mathcal{W}_1} [I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1} V_1^*] \eta,$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Finally, from $\tilde{\Pi}_1^* M_z^{\mathcal{W}_1} = M_{\Phi_1} \tilde{\Pi}_1^*$ and $\tilde{\Pi}_1^* \eta_1 = \eta_1$ for all $\eta_1 \in \mathcal{W}_1$, it follows that $\tilde{\Pi}_1^*(z^m \eta_1) = M_{\Phi_1}^m \eta_1$ for all $m \geq 0$, and hence

$$\tilde{\Pi}_1^*(\mathbb{S}(\cdot, w)\eta_1) = (I_{\mathcal{W}} - \Phi_1(z)\bar{w})^{-1} \eta_1,$$

for all $w \in \mathbb{D}$ and $\eta_1 \in \mathcal{W}_1$.

We summarize the above observations in the following theorem.

Theorem 5.1. *Let* (V_1, V_2) *be a pair of commuting isometries on* $\mathcal{H}$. *Let* $i, j \in \{1, 2\}$ *and* $i \neq j$. *If* V_i *is a pure isometry, then*

$$\tilde{\Pi}_i = \Pi_{V_i} \Pi_V^* \in \mathcal{B}(H_{\mathcal{W}}^2(\mathbb{D}), H_{\mathcal{W}_i}^2(\mathbb{D})),$$

is a unitary operator,

$$\tilde{\Pi}_i M_z^{\mathcal{W}} = M_{z\Theta_{V_j}} \tilde{\Pi}_i, \ \tilde{\Pi}_i^* M_z^{\mathcal{W}_i} = M_{\Phi_i} \tilde{\Pi}_i^*,$$

and

$$\tilde{\Pi}_i(\mathbb{S}(\cdot, w)\eta) = (I_{\mathcal{W}_i} - \bar{w}z\Theta_{V_j}(z))^{-1} P_{\mathcal{W}_i} [I_{\mathcal{H}} + z(I - zV_i^*)^{-1} V_i^*] \eta,$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$, where

$$\Theta_{V_j}(z) = P_{\mathcal{W}_i}(I_{\mathcal{H}} - zV_i^*)^{-1}V_j|_{\mathcal{W}_i}$$

for all $z \in \mathbb{D}$. Moreover

$$\tilde{\Pi}_i^*(\mathbb{S}(\cdot, w)\eta_i) = (I_{\mathcal{W}} - \Phi_i(z)\bar{w})^{-1}\eta_i,$$

for all $w \in \mathbb{D}$ and $\eta_i \in \mathcal{W}_i$.

Note that the inner multipliers $\Theta_{V_i} \in H^\infty_{\mathcal{B}(\mathcal{W}_j)}(\mathbb{D})$ above satisfy the following equalities:

$$\Pi_{V_j}V_i = M_{\Theta_{V_i}}\Pi_{V_j}.$$

Now let (V_1, V_2) be a pair of commuting isometries such that both V_1 and V_2 are pure isometries. The above result leads to an analytic representation of such pairs.

Corollary 5.2. *Let (V_1, V_2) be a pair of commuting pure isometries on a Hilbert space $\mathcal{H}$. If (M_{Φ_1}, M_{Φ_2}) is the BCL representation corresponding to (V_1, V_2), then M_{Φ_1} and M_{Φ_2} are pure isometries,*

$$\tilde{\Pi}_1 M_{\Phi_2} = M_{\Theta_{V_2}}\tilde{\Pi}_1, \ \tilde{\Pi}_2 M_{\Phi_1} = M_{\Theta_{V_1}}\tilde{\Pi}_2,$$

$\tilde{\Pi} = \tilde{\Pi}_2\tilde{\Pi}_1^* : H^2_{\mathcal{W}_1}(\mathbb{D}) \to H^2_{\mathcal{W}_2}(\mathbb{D})$ *is a unitary operator, and*

$$\tilde{\Pi}M_z^{\mathcal{W}_1} = M_{\Theta_{V_1}}\tilde{\Pi} \text{ and } \tilde{\Pi}M_{\Theta_{V_2}} = M_z^{\mathcal{W}_2}\tilde{\Pi}.$$

Moreover, for each $w \in \mathbb{D}$ and $\eta_j \in \mathcal{W}_j$, $j = 1, 2$,

$$\tilde{\Pi}(\mathbb{S}(\cdot, w)\eta_1) = (I_{\mathcal{W}_2} - \bar{w}\Theta_{V_1}(z))^{-1}P_{\mathcal{W}_2}(I_{\mathcal{H}} - zV_2^*)^{-1}\eta_1,$$

and

$$\tilde{\Pi}^*(\mathbb{S}(\cdot, w)\eta_2) = (I_{\mathcal{W}_1} - \bar{w}\Theta_{V_2}(z))^{-1}P_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}\eta_2.$$

Proof. A repeated application of Theorem 5.1 yields

$$\begin{aligned}
\tilde{\Pi}_1 M_{\Phi_2} &= \tilde{\Pi}_1 M_{\Phi_1}^*(M_{\Phi_1}M_{\Phi_2})\\
&= \tilde{\Pi}_1 M_{\Phi_1}^* M_z^{\mathcal{W}}\\
&= (M_z^{\mathcal{W}_1})^*\tilde{\Pi}_1 M_z^{\mathcal{W}}\\
&= (M_z^{\mathcal{W}_1})^* M_{z\Theta_{V_2}}\tilde{\Pi}_1,
\end{aligned}$$

that is, $\tilde{\Pi}_1 M_{\Phi_2} = M_{\Theta_{V_2}}\tilde{\Pi}_1$ and similarly $\tilde{\Pi}_2 M_{\Phi_1} = M_{\Theta_{V_1}}\tilde{\Pi}_2$. For $\eta_1 \in \mathcal{W}_1$, we have $\Pi_{V_2}\eta_1 = P_{\mathcal{W}_2}(I_{\mathcal{H}} - zV_2^*)^{-1}\eta_1$. Since $\tilde{\Pi}_1^*\eta_1 = \eta_1$ and $\Pi_V^*\eta_1 = \eta_1$, it follows that

$$\tilde{\Pi}\eta_1 = \tilde{\Pi}_2\eta_1 = \Pi_{V_2}\Pi_V^*\eta_1 = \Pi_{V_2}\eta_1,$$

that is $\tilde{\Pi}\eta_1 = P_{\mathcal{W}_2}(I_{\mathcal{H}} - zV_2^*)^{-1}\eta_1$. Now using the identity $\tilde{\Pi}(z\eta_1) = M_{\Theta_{V_1}}\tilde{\Pi}\eta_1$, we have

$$\tilde{\Pi}(z^m\eta_1) = \Theta_{V_1}(z)^m P_{\mathcal{W}_2}(I_{\mathcal{H}} - zV_2^*)^{-1}\eta_1,$$

for all $m \geq 0$ and $\eta_1 \in \mathcal{W}_1$. Finally $\mathbb{S}(\cdot, w)\eta_1 = \sum_{m=0}^{\infty} \bar{w}^m z^m \eta_1$ gives

$$\tilde{\Pi}(\mathbb{S}(\cdot, w)\eta_1) = (I_{\mathcal{W}_2} - \bar{w}\Theta_{V_1}(z))^{-1} P_{\mathcal{W}_2}(I_{\mathcal{H}} - zV_2^*)^{-1}\eta_1.$$

The final equality of the corollary follows from the equality

$$\tilde{\Pi}^*(z^m \eta_2) = \Theta_{V_2}(z)^m(\tilde{\Pi}^* \eta_2) = \Theta_{V_2}(z)^m P_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}\eta_2,$$

for all $m \geq 0$ and $\eta_2 \in \mathcal{W}_2$. This concludes the proof.   ∎

In the final section, we will connect the analytic descriptions of $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ as in Theorem 5.1 with the classical notion of the Sz.-Nagy and Foias characteristic functions of contractions on Hilbert spaces [NF].

6. Defect Operators

Throughout this section, we will mostly work on general (not necessarily pure) pairs of commuting isometries. Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. The *defect operator* $C(V_1, V_2)$ of (V_1, V_2) (cf. [HQY]) is defined as the self-adjoint operator

$$C(V_1, V_2) = I - V_1 V_1^* - V_2 V_2^* + V_1 V_2 V_1^* V_2^*.$$

Recall from Section 3 that given a pair of commuting isometries (V_1, V_2), we write $V = V_1 V_2$, and denote by

$$\mathcal{W}_j = \mathcal{W}(V_j) = \ker V_j^* = \mathcal{H} \ominus V_j \mathcal{H},$$

the wandering subspace for V_j, $j = 1, 2$. The wandering subspace for V is denoted by $\mathcal{W}$. Finally, we recall that (see Lemma 3.1) $\mathcal{W} = \mathcal{W}_1 \oplus V_1 \mathcal{W}_2 = V_2 \mathcal{W}_1 \oplus \mathcal{W}_2$. This readily implies

(6.1) $$P_{\mathcal{W}} = P_{\mathcal{W}_1} \oplus P_{V_1 \mathcal{W}_2} = P_{V_2 \mathcal{W}_1} \oplus P_{\mathcal{W}_2}.$$

The following lemma is well known to the experts, but for the sake of completeness we provide a proof of the statement.

Lemma 6.1. *Let (V_1, V_2) be a commuting pair of isometries on $\mathcal{H}$. Then $\mathcal{H}_s(V)$ and $\mathcal{H}_u(V)$ are V_j-reducing subspaces,*

$$\mathcal{H}_s(V_j) \subseteq \mathcal{H}_s(V), \text{ and } \mathcal{H}_u(V_j) \supseteq \mathcal{H}_u(V),$$

for all $j = 1, 2$.

Proof. For the first part we only need to prove that $\mathcal{H}_s(V)$ is a V_1-reducing subspace. Note that since (see Lemma 3.1) $V_1 \mathcal{W} \subseteq \mathcal{W} \oplus V \mathcal{W}$, it follows that

$$V_1 V^m \mathcal{W} \subseteq V^m(\mathcal{W} \oplus V \mathcal{W}) \subseteq \mathcal{H}_s(V),$$

for all $m \geq 0$. This clearly implies that $V_1 \mathcal{H}_s(V) \subseteq \mathcal{H}_s(V)$. On the other hand, since $V_1^* \mathcal{W} = \mathcal{W}_2 \subseteq \mathcal{W}$ and

$$V_1^* V^m \mathcal{W} = V^{m-1}(V_2 \mathcal{W}) \subseteq V^{m-1}(\mathcal{W} \oplus V\mathcal{W}),$$

it follows that $V_1^* \mathcal{H}_s(V) \subseteq \mathcal{H}_s(V)$. To prove the second part of the statement, it is enough to observe that

$$V^m \mathcal{H} = V_1^m(V_2^m \mathcal{H}) = V_2^m(V_1^m \mathcal{H}) \subseteq V_1^m \mathcal{H}, V_2^m \mathcal{H},$$

for all $m \geq 0$, and as $n \to \infty$

$$V_1^{*n} h \to 0, \text{ or } V_2^{*n} h \to 0 \Rightarrow V^{*n} h \to 0,$$

for any $h \in \mathcal{H}$. This concludes the proof of the lemma.   ■

The following characterizations of doubly commuting isometries will prove important in the sequel.

Lemma 6.2. *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*
(i) (V_1, V_2) is doubly commuting.
(ii) $V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1$.
(iii) $V_1 \mathcal{W}_2 \subseteq \mathcal{W}_2$.

Proof. Since (i) implies (ii) and (iii), by symmetry we only need to show that (ii) implies (i). Let $V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1$. Let $\mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V)$ be the Wold-von Neumann orthogonal decomposition of V (see Theorem 2.1). Then $\mathcal{H}_s(V)$ and $\mathcal{H}_u(V)$ are joint (V_1, V_2)-reducing subspaces, and the pair $(V_1|_{\mathcal{H}_u(V)}, V_2|_{\mathcal{H}_u(V)})$ on $\mathcal{H}_u$ is doubly commuting, because $V_j|_{\mathcal{H}_u(V)}, j = 1, 2$, are unitary operators, by Lemma 6.1. Now it only remains to prove that $V_1^* V_2 = V_2 V_1^*$ on $\mathcal{H}_s(V)$. Since

$$(V_1^* V_2 - V_2 V_1^*) V^m = V_1^* V^m V_2 - V_2 V_1^* V^m = V^{m-1} V_2^2 - V_2^2 V^{m-1} = 0,$$

it follows that $V_1^* V_2 - V_2 V_1^* = 0$ on $V^m \mathcal{W}$ for all $m \geq 1$. In order to complete the proof we must show that $V_1^* V_2 = V_2 V_1^*$ on $\mathcal{W}$. To this end, let $\eta = \eta_1 \oplus V_1 \eta_2 \in \mathcal{W}$ for some $\eta_1 \in \mathcal{W}_1$ and $\eta_2 \in \mathcal{W}_2$. Then

$$V_1^* V_2(\eta_1 \oplus V_1 \eta_2) = V_1^* V_2 \eta_1 + V_1^* V_2 V_1 \eta_2 = V_2 \eta_2,$$

as $V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1$, and on the other hand

$$V_2 V_1^*(\eta_1 \oplus V_1 \eta_2) = V_2 V_1^* \eta_1 + V_2 V_1^* V_1 \eta_2 = V_2 \eta_2.$$

This completes the proof.   ■

The key of our geometric approach is the following simple representation of defect operators.

Lemma 6.3.

$$C(V_1, V_2) = P_{\mathcal{W}_1} - P_{V_2 \mathcal{W}_1} = P_{\mathcal{W}_2} - P_{V_1 \mathcal{W}_2}.$$

Proof. The result readily follows from (6.1) and

$$C(V_1, V_2) = (I - V_1 V_1^*) + (I - V_2 V_2^*) - (I - VV^*)$$
$$= P_{\mathcal{W}_1} + P_{\mathcal{W}_2} - P_{\mathcal{W}}.$$

$\blacksquare$

The final ingredient to our analysis is the fringe operator F_2. The notion of fringe operators plays a significant role in the study of joint shift-invariant closed subspaces of the Hardy space over $\mathbb{D}^2$ (see the discussion at the beginning of Section 5). Given a pair of commuting isometries (V_1, V_2) on $\mathcal{H}$, the *fringe operators* $F_1 \in \mathcal{B}(\mathcal{W}_2)$ and $F_2 \in \mathcal{B}(\mathcal{W}_1)$ are defined by

$$F_j = P_{\mathcal{W}_i} V_j|_{\mathcal{W}_i} \qquad (i \neq j).$$

Of particular interest to us are the isometric fringe operators. Note that

$$F_2^* F_2 = P_{\mathcal{W}_1} V_2^* P_{\mathcal{W}_1} V_2|_{\mathcal{W}_1}.$$

Lemma 6.4. *The fringe operator F_2 on $\mathcal{W}_1$ is an isometry if and only if $V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1$.*

Proof. As $I_{\mathcal{W}_1} - F_2^* F_2 = I_{\mathcal{W}_1} - P_{\mathcal{W}_1} V_2^* P_{\mathcal{W}_1} V_2|_{\mathcal{W}_1}$, (6.1) implies that

$$I_{\mathcal{W}_1} - F_2^* F_2 = P_{\mathcal{W}_1} V_2^* P_{V_1 \mathcal{W}_2} V_2|_{\mathcal{W}_1}.$$

Then $F_2^* F_2 = I_{\mathcal{W}_1}$ if and only if $P_{V_1 \mathcal{W}_2} V_2|_{\mathcal{W}_1} = 0$, or, equivalently, if and only if $V_2 \mathcal{W}_1 \perp V_1 \mathcal{W}_2 = \mathcal{W}_1^\perp$, by Lemma 3.1. This completes the proof. $\blacksquare$

Therefore, the fringe operator F_2 is an isometry if and only if the pair (V_1, V_2) is doubly commuting.

We are now ready to formulate a generalization of Theorem 3.4 in [HQY] by He, Qin and Yang. Here we do not assume that (V_1, V_2) is pure.

Theorem 6.5. *Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$. Then the following are equivalent:*
(a) $C(V_1, V_2) \geq 0$.
(b) $V_2 \mathcal{W}_1 \subseteq \mathcal{W}_1$.
(c) (V_1, V_2) is doubly commuting.
(d) $C(V_1, V_2)$ is a projection.
(e) The fringe operator F_2 is an isometry.

Proof. The equivalences of (a) and (b), (b) and (c), and (b) and (e) are given in Lemma 6.3, Lemma 6.2 and Lemma 6.4, respectively. The implication (c) implies (d) follows from

$$C(V_1, V_2) = P_{\mathcal{W}_1} P_{\mathcal{W}_2} = P_{\mathcal{W}_2} P_{\mathcal{W}_1}.$$

Clearly (d) implies (a). This completes the proof. $\qquad\blacksquare$

We now prove that for a large class of pairs of commuting isometries negative defect operator always implies the zero defect operator.

Theorem 6.6. *Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$. Suppose that V_1 or V_2 is pure. Then $C(V_1, V_2) \leq 0$ if and only if $C(V_1, V_2) = 0$.*

Proof. With out loss of generality assume that V_2 is pure. If $C(V_1, V_2) \leq 0$, then by Lemma 6.3, we have $P_{\mathcal{W}_1} \leq P_{V_2 \mathcal{W}_1}$, or, equivalently

$$\mathcal{W}_1 \subseteq V_2 \mathcal{W}_1,$$

and hence

$$\mathcal{W}_1 \subseteq V_2{}^m \mathcal{W}_1 \subseteq V_2{}^m \mathcal{H},$$

for all $m \geq 0$. Therefore

$$\mathcal{W}_1 = \bigcap_{m=0}^{\infty} V_2{}^m \mathcal{W}_1 \subseteq \bigcap_{m=0}^{\infty} V_2{}^m \mathcal{H} = \{0\},$$

as V_2 is pure. Hence $\mathcal{W}_1 = \{0\}$ and $V_2 \mathcal{W}_1 = \{0\}$. This gives $C(V_1, V_2) = P_{\mathcal{W}_1} - P_{V_2 \mathcal{W}_1} = 0$. $\qquad\blacksquare$

The same conclusion holds if we allow $\dim \mathcal{W}_j < \infty$ for some $j \in \{1, 2\}$.

Theorem 6.7. *Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$. Suppose that $\dim \mathcal{W}_j < \infty$ for some $j \in \{1, 2\}$. Then $C(V_1, V_2) \leq 0$ if and only if $C(V_1, V_2) = 0$.*

Proof. We may suppose that $\dim \mathcal{W}_1 < \infty$. Let $C(V_1, V_2) \leq 0$. Since $\mathcal{W}_1 \subseteq V_2 \mathcal{W}_1$ and V_2 is an isometry, it follows that

$$\mathcal{W}_1 = V_2 \mathcal{W}_1.$$

Hence $C(V_1, V_2) = P_{\mathcal{W}_1} - P_{V_2 \mathcal{W}_1} = 0$. This completes the prove. $\qquad\blacksquare$

The same conclusion also holds for positive defect operators.

7. Concluding Remarks

As pointed out in the introduction, a general theory for pairs of commuting isometries is mostly unknown and unexplored (however, see Popovici [P]). In comparison, we would like to add that a great deal is known about the structure of pairs (and even of n-tuples) of commuting isometries with finite rank defect operators (see [BKS], [BKPS1], [BKPS2]). A complete classification result is also known for n-tuples of doubly commuting isometries (cf. [GS], [S], [JS]). It is now natural to ask whether the present results for pure pairs of commuting isometries can be extended to arbitrary pairs of commuting isometries (see [D], [GG] and [GS] for closely related results). Another relevant question is to analyze the joint shift invariant subspaces of the Hardy space over the unit bidisc [ACD] from our analytic and geometric point of views. More detailed discussion on these issues will be given in forthcoming papers.

Also we point out that some of the results of this paper can be extended to n-tuples of commuting isometries and will be discussed in a future paper.

We conclude this paper by inspecting a connection between the Sz.-Nagy and Foias characteristic functions of contractions on Hilbert spaces [NF] and the analytic representations of $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ as described in Theorem 5.1. Let T be a contraction on a Hilbert space $\mathcal{H}$. The *defect operators* of T, denoted by D_{T^*} and D_T, are defined by

$$D_{T^*} = (I - TT^*)^{1/2}, \quad D_T = (I - T^*T)^{1/2}.$$

The defect spaces, denoted by $\mathcal{D}_{T^*}$ and $\mathcal{D}_T$, are the closure of the ranges of D_{T^*} and D_T, respectively. The *characteristic function* [NF] of the contraction T is defined by

$$\theta_T(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T]|_{\mathcal{D}_T} \qquad (z \in \mathbb{D}).$$

It follows that $\theta_T \in H^\infty_{\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})}(\mathbb{D})$ [NF]. The characteristic function is a complete unitary invariant for the class of completely non-unitary contractions. This function is also closely related to the Beurling-Lax-Halmos inner functions for shift invariant subspaces of vector-valued Hardy spaces. For a more detailed discussion of the theory and applications of characteristic functions we refer to the monograph by Sz.-Nagy and Foias [NF].

Now let us return to the study of pairs of commuting isometries. Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$. We compute

$$
\begin{aligned}
P_{\mathcal{W}_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} &= [P_{\mathcal{W}_1} + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} \\
&= [I_{\mathcal{H}} - V_1V_1^* + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} \\
&= I_{\mathcal{W}} + [-V_1 + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}]V_1^*|_{\mathcal{W}}.
\end{aligned}
$$

Since $V_1^*\mathcal{W} = \mathcal{W}_2$, it follows that

$$
[-V_1 + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}]V_1^*|_{\mathcal{W}} = [-V_1 + zD_{V_1^*}(I_{\mathcal{H}} - zV_1^*)^{-1}D_{V_2^*}]|_{\mathcal{D}_{V_2^*}}(V_1^*|_{\mathcal{W}}).
$$

Therefore, setting

(7.1)
$$
\theta_{V_1,V_2}(z) = [-V_1 + zD_{V_1^*}(I_{\mathcal{H}} - zV_1^*)^{-1}D_{V_2^*}]|_{\mathcal{D}_{V_2^*}},
$$

for $z \in \mathbb{D}$, we have

$$
P_{\mathcal{W}_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} = I_{\mathcal{W}} + \theta_{V_1,V_2}(z)V_1^*|_{\mathcal{W}},
$$

for all $z \in \mathbb{D}$. Therefore, if V_1 is a pure isometry, then the formula for $\tilde{\Pi}_1$ in Theorem 5.1(i) can be expressed as

$$
\tilde{\Pi}_1(\mathbb{S}(\cdot, w)\eta) = (I_{\mathcal{W}_1} - \bar{w}\Theta_{V_2}(z))^{-1}P_{\mathcal{W}_1}[I_{\mathcal{W}} + \theta_{V_1,V_2}(z)V_1^*|_{\mathcal{W}}]\eta.
$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Similarly, if V_2 is a pure isometry, then the formula for $\tilde{\Pi}_2$ in Theorem 5.1 (ii) can be expressed as

$$
\tilde{\Pi}_2(\mathbb{S}(\cdot, w)\eta) = (I_{\mathcal{W}_2} - \bar{w}\Theta_{V_1}(z))^{-1}P_{\mathcal{W}_2}[I_{\mathcal{W}} + \theta_{V_2,V_1}(z)V_2^*|_{\mathcal{W}}]\eta,
$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$, where

(7.2)
$$
\theta_{V_2,V_1}(z) = [-V_2 + zD_{V_2^*}(I_{\mathcal{H}} - zV_2^*)^{-1}D_{V_1^*}]|_{\mathcal{D}_{V_1^*}},
$$

for all $z \in \mathbb{D}$.

It is easy to see that $\theta_{V_i,V_j}(z) \in \mathcal{B}(\mathcal{W}_j, \mathcal{W})$ for all $z \in \mathbb{D}$ and $i \neq j$.

Note that since the defect operator $D_{V_j} = 0$, the characteristic function θ_{V_j} of V_j, $j = 1, 2$, is the zero function. From this point of view, it is expected that the pair of analytic invariants $\{\theta_{V_i,V_j} : i \neq j\}$ will provide more information about pairs of commuting isometries.

Subsequent theory for pairs of commuting contractions and a more detailed connection between pairs of commuting pure isometries (V_1, V_2) and the analytic invariants $\{\theta_{V_i,V_j} : i \neq j\}$ as defined in (7.1) and (7.2) will be exhibited in more details in future occasion.

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